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J. Math. Anal. Appl. 278 (2003) 194–202

Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS

www.elsevier.com/locate/jmaa

# Oscillation and global asymptotic stability in a discrete epidemic model ☆

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Received 13 October 2000

Submitted by R.P. Agarwal

**Abstract**

In this paper, we study the oscillation, global asymptotic stability, and other properties of the positive solutions of the difference equation

$$x_{n+1} = \left(1 - \sum_{j=0}^{k-1} x_{n-j}\right)\left(1 - e^{-Ax_n}\right),$$

where $A \in (0, \infty)$, $k \in \{2, 3, \ldots\}$, and the initial values $x_{-k+1}, x_{-k+2}, \ldots, x_0$ are arbitrary real positive numbers such that $\sum_{j=0}^{k-1} x_{-j} < 1$.


*Keywords:* Nonlinear difference equations; Oscillation; Global asymptotic stability

## 1. Introduction

In the monograph of Kocic and Ladas [1], they give two research projects (see [1, p. 169]).

☆ Research supported by the Distinguished Expert Foundation of the Naval Aeronautical Engineering Academy.
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0022-247X/03/$ – see front matter 
doi:10.1016/S0022-247X(02)00717-5

**Research project 6.71.** Investigate the oscillatory behavior, the global asymptotic stability and periodic character of the solutions of the following equation:

$$x_{n+1} = \left(1 - \sum_{j=0}^{k-1} x_{n-j}\right)(1 - e^{-Ax_n}); \tag{1}$$

and

**Research project 6.72.** How should we choose the initial conditions of Eq. (1) so that the solutions remain positive for all $n = 0, 1, \ldots$?

To this end we consider Eq. (1), where $A \in (0, \infty)$, $k \in \{2, 3, \ldots\}$, and the initial values $x_{-k+1}, x_{-k+2}, \ldots, x_0$ are arbitrary real positive numbers such that $\sum_{j=0}^{k-1} x_{-j} < 1$.

The aim of this paper is to investigate the oscillation and global asymptotic stability of the nonnegative equilibrium of Eq. (1).

## 2. Some lemmas

**Lemma 2.1.** *Assume that* $0 < A \leqslant 1$. *Then the equation*

$$f(x) = (1 - x)(1 - e^{-Ax}) = x \tag{2}$$

*has a unique nonnegative root* $x = 0$, *and* $f(x) < x$ *for* $0 < x < 1$.

The proof of this lemma is very easy, we omit it.

**Lemma 2.2.** (i) *Assume that* $0 < A \leqslant 1$. *Then the equation*

$$g(x) = (1 - kx)(1 - e^{-Ax}) = x \tag{3}$$

*has a unique nonnegative root* $x = 0$ *for* $0 \leqslant x < 1/k$;

(ii) *Assume that* $A > 1$. *Then Eq.* (3) *has a unique positive root* $\bar{x}$ *for* $0 < x < 1/k$, *where* $\bar{x} < 1/(k+1)$.

**Proof.** From $g(x) = (1 - kx)(1 - e^{-Ax})$, we get $g'(x) = -k + e^{-Ax}(A - kAx + k)$ and $g''(x) = -Ae^{-Ax}(2k + A - kAx) < 0$ for $0 \leqslant x < 1/k$. Hence, $g(x)$ is a convex function.

(i) Noting that $g(0) = 0$ and $g'(x) < g'(0) = A \leqslant 1$, by the properties of $g(x)$ we obtain that Eq. (3) has a unique nonnegative root $x = 0$ for $0 \leqslant x < 1/k$.

(ii) Observing that $g(0) = 0$ and $g'(x) < g'(0) = A > 1$, by the properties of $g(x)$ we obtain that Eq. (3) has a unique nonnegative root $x = \bar{x}$, where $\bar{x} < 1/(k+1)$ for $0 \leqslant x < 1/k$.

Hence, this completes the proof. □

**Lemma 2.3.** *Assume that* $k \in \{1, 2, \ldots\}$. *Then*

$$1 + \frac{1}{k} > \exp\left(\frac{3k + 1 - \sqrt{5k^2 + 2k + 1}}{2k(k+1)}\right) + \frac{3k + 1 - \sqrt{5k^2 + 2k + 1}}{2k(k+1)}. \tag{4}$$

**Proof.** It is easy to see that (4) holds for $k \in \{1, 2, 3\}$. So we need only to prove that (4) holds for $k \in \{4, 5, \ldots\}$.

Set

$$R(k) = \exp\left(\frac{3k+1-\sqrt{5k^2+2k+1}}{2k(k+1)}\right) + \frac{3k+1-\sqrt{5k^2+2k+1}}{2k(k+1)}.$$

Then

$$R(k) = \exp\left(\frac{2}{3k+1+\sqrt{5k^{2+2k+1}}}\right) + \frac{2}{3k+1+\sqrt{5k^2+2k+1}}.$$

Observing that

$$0 < \frac{2}{3k+1+\sqrt{5k^2+2k+1}} < \frac{1}{6} \quad \text{for } k \in \{4, 5, \ldots\},$$

by the properties of

$$H(x) = e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots,$$

we have

$$\begin{aligned} R(k) < 1 &+ \frac{2}{3k+1+\sqrt{5k^2+2k+1}} + \left(\frac{2}{3k+1+\sqrt{5k^2+2k+1}}\right)^2 \\ &+ \frac{2}{3k+1+\sqrt{5k^2+2k+1}} \end{aligned}$$

and

$$\begin{aligned} 1 + \frac{1}{k} - \Bigg[1 &+ \frac{2}{3k+1+\sqrt{5k^2+2k+1}} + \left(\frac{2}{3k+1+\sqrt{5k^2+2k+1}}\right)^2 \\ &+ \frac{2}{3k+1+\sqrt{5k^2+2k+1}}\Bigg] \\ = \frac{2k^2+2+2(k+1)\sqrt{5k^2+2k+1}}{k(3k+1+\sqrt{5k^2+2k+1})^2} > 0. \end{aligned}$$

Then (4) holds for $k \in \{1, 2, \ldots\}$. Thus the proof is complete. □

**Lemma 2.4.** *The equation*

$$\begin{aligned} 1 + \frac{A}{k} = \exp&\left(\frac{(2k+1)A + k - \sqrt{A^2 + 2k(2k+1)A + k^2}}{2k(2k+1)}\right) \\ &+ \frac{(2k+1)A + k - \sqrt{A^2 + 2k(2k+1)A + k^2}}{2k(2k+1)} \end{aligned} \tag{5}$$

*for $k \in \{1, 2, \ldots\}$ has a unique root $A^*$ for $A > 1$. Moreover,*

$$1+\frac{A}{k} > \exp\left(\frac{(2k+1)A+k-\sqrt{A^2+2k(2k+1)A+k^2}}{2k(2k+1)}\right) + \frac{(2k+1)A+k-\sqrt{A^2+2k(2k+1)A+k^2}}{2k(2k+1)} \tag{6}$$

*holds for* $1 < A < A^*$, *and*

$$1+\frac{A}{k} < \exp\left(\frac{(2k+1)A+k-\sqrt{A^2+2k(2k+1)A+k^2}}{2k(2k+1)}\right) + \frac{(2k+1)A+k-\sqrt{A^2+2k(2k+1)A+k^2}}{2k(2k+1)} \tag{7}$$

*holds for* $A > A^*$.

**Proof.** Set $H(A) = 1 + A/k$ and

$$G(A) = \exp\left(\frac{(2k+1)A+k-\sqrt{A^2+2k(2k+1)A+k^2}}{2k(2k+1)}\right) + \frac{(2k+1)A+k-\sqrt{A^2+2k(2k+1)A+k^2}}{2k(2k+1)}.$$

Then

$$H(1) = 1+\frac{1}{k} \quad \text{and}$$
$$G(1) = \exp\left(\frac{3k+1-\sqrt{5k^2+2k+1}}{2k(k+1)}\right) + \frac{3k+1-\sqrt{5k^2+2k+1}}{2k(k+1)}.$$

By Lemma 2.3 we know that $H(1) > G(1)$. Noting that $H'(A) = 1/k$,

$$G'(A) = \frac{1}{2k(k+1)}\left[(2k+1) - \frac{A+k(2k+1)}{\sqrt{A^2+2k(2k+1)A+k^2}}\right] \times \left[\exp\left(\frac{(2k+1)A+k-\sqrt{A^2+2k(2k+1)A+k^2}}{2k(k+1)}\right)+1\right]$$

and

$$G''(A) = \exp\left(\frac{(2k+1)A+k-\sqrt{A^2+2k(2k+1)A+k^2}}{2k(k+1)}\right) \times \left[\frac{1}{2k(k+1)}\left((2k+1) - \frac{A+k(2k+1)}{\sqrt{A^2+2k(2k+1)A+k^2}}\right)\right]^2 + \exp\left(\frac{(2k+1)A+k-\sqrt{A^2+2k(2k+1)A+k^2}}{2k(k+1)}\right) \times \frac{2k^2}{(A^2+2k(2k+1)A+k^2)^{3/2}} > 0.$$

Hence $G(A)$ is a convex function. Thus by the properties of $G(A)$, we obtain that Eq. (5) has a unique positive solution $A^*$; (6) and (7) hold. Thus the proof is complete. □

**Lemma 2.5.** *Assume that* $A = A^*$. *Then Eq.* (3) *has the same root as the following equation*:

$$g'(x) = \big[(1-kx)(1-e^{-Ax})\big]' = 0, \tag{8}$$

*i.e.,*

$$\bar{x} = x^* = \frac{(2k+1)A^* + k - \sqrt{A^* + 2A^*(2k+1)k + k^2}}{2A^*k(k+1)},$$

*where* $\bar{x}$ *or* $x^*$ *is the positive root of Eq.* (3) *or Eq.* (8), *respectively.*

**Proof.** From (3) and (8), we have, respectively, that

$$e^{-Ax} = \frac{1-(k+1)x}{1-kx}$$

and

$$e^{-Ax} = \frac{k}{A - kAx + k}. \tag{9}$$

Thus

$$\frac{1-(k+1)x}{1-kx} = \frac{k}{A-kAx+k}.$$

It follows

$$x = \frac{(2k+1)A + k - \sqrt{A^2 + 2A(2k+1)k + k^2}}{2Ak(k+1)}. \tag{10}$$

Substituting (10) into (8), we obtain

$$1 + \frac{A}{k} = \exp\left(\frac{(2k+1)A + k - \sqrt{A^2 + 2k(2k+1)A + k^2}}{2k(2k+1)}\right) + \frac{(2k+1)A + k - \sqrt{A^2 + 2k(2k+1)A + k^2}}{2k(2k+1)}. \tag{11}$$

Hence, (11) holds for $A = A^*$ from Lemma 2.4.

By virtue of Lemma 2.2 and the properties of $g(x)$, Eq. (3) and Eq. (8) have the same root, i.e.,

$$\bar{x} = x^* = \frac{(2k+1)A^* + k - \sqrt{A^{*2} + 2k(2k+1)A^* + k^2}}{2k(k+1)A^*}.$$

Therefore, the proof is complete. □

**Lemma 2.6.** *Assume that* $\bar{x}$ *is the root of the Eq.* (3) *and* $x^*$ *is the root of the Eq.* (8). *Then*

(i) $\bar{x} < x^*$ *holds for* $1 < A < A^*$;

(ii) *$\bar{x} > x^*$ holds for $A > A^*$;*
(iii) *If $1 < A < A^*$, then $g(x) > x$ for $0 < x < \bar{x}$, and $g(x) < x$ for $\bar{x} < x < 1/k$;*
(iv) *If $A = A^*$, then $g(x) > x$ for $0 < x < \bar{x}$, and $g(x) < \bar{x} < x$ for $\bar{x} < x < 1/k$;*
(v) *If $A > A^*$, then there exists $\bar{x}^*$ which is a positive number and $0 < \bar{x}^* < x^*$ such that $g(\bar{x}^*) = g(\bar{x})$, and $g(x) > \bar{x}$ holds for $\bar{x}^* < x < \bar{x}$, $g(x) > x$ holds for $0 < x < \bar{x}$, and $g(x) < \bar{x} < x$ holds for $\bar{x} < x < 1/k$.*

**Proof.** (i) Set

$$a = \frac{(2k+1)A + k - \sqrt{A^2 + 2A(2k+1)k + k^2}}{2Ak(k+1)}.$$

Then we have $g'(a) = -k + e^{-Aa}(A - kAa + k)$.

From (6) we obtain $g'(a) > -k + k = 0$. Thus by the properties of $g(x)$, we have

$$a < x^*. \tag{12}$$

From (6) and Eq. (3), we get

$$g(a) - a = (1 - ka)(1 - e^{-Aa}) - a < 0.$$

Hence $g(a) < a$. This follows

$$\bar{x} < a. \tag{13}$$

Therefore, $\bar{x} < x^*$ holds from (12) and (13). This completes the proof of part (i).

The proof of (ii) is just the same as (i), we omit it. Moreover, the proofs of (iii), (iv), and (v) can be easily proved, we also omit them.

Thus we complete the proof of this lemma. □

**Lemma 2.7.** *Assume that $H(x) = (1 - e^{-Ax})/x$, where $0 < x$ and $A > 1$. Then $H'(x) < 0$. It follows that*

$$\frac{1 - e^{-Ax_1}}{1 - e^{-Ax_2}} < \frac{x_1}{x_2} \quad \textit{for } 0 < x_2 < x_1.$$

**Proof.** Observing that $H'(x) = ((Ax+1)e^{-Ax} - 1)/x^2$, setting $G(x) = (Ax+1)e^{-Ax} - 1$, and noting that $G(0) = 0$ and $G'(x) = -A^2xe^{-Ax} < 0$, we have $G(x) < G(0) = 0$ for $0 < x$. Thus $H'(x) < 0$. It follows that

$$\frac{1 - e^{-Ax_1}}{1 - e^{-Ax_2}} < \frac{x_1}{x_2} \quad \text{for } 0 < x_2 < x_1.$$

This completes the proof. □

## 3. Main results

**Theorem 3.1.** *Assume that $0 < x_{-k+1} + x_{-k+2} + \cdots + x_{-1} + x_0 < 1$ and $x_i \geqslant 0$, where $i \in \{-k+1, -k+2, \ldots, -1, 0\}$. Then every solution of Eq.* (1) *remains positive for all $n = 1, 2, \ldots$.*

**Proof.** From Eq. (1) we obtain $0 < x_1 < 1 - \sum_{j=0}^{k-1} x_{0-j} < 1$. It follows that $0 < \sum_{j=0}^{k} x_{1-j} < 1$. Thus $0 < x_1 + x_0 + x_{-1} + \cdots + x_{-k+2} < 1$.

By induction we know that $0 < x_2 < 1$ and $0 < x_2 + x_1 + x_0 + x_{-1} + \cdots + x_{-k+3} < 1$. Therefore, every solution of Eq. (1) remains positive for all $n = 1, 2, \ldots$. Thus the proof is complete. □

**Theorem 3.2.** *Assume that* $0 < A \leqslant 1$. *Then every positive solution of Eq.* (1) *is strictly decreasing and converges to* 0.

**Proof.** From Eq. (1) we have

$$x_{n+1} = \left(1 - \sum_{j=0}^{k-1} x_{n-j}\right)(1 - e^{-Ax_n}) < (1 - x_n)(1 - e^{-Ax_n}) \quad \text{for } n \geqslant 0.$$

By Lemma 2.1, we obtain that $x_{n+1} < x_n$ for $0 \leqslant n$. Thus the solution $\{x_n\}_{n=0}^{\infty}$ is strictly decreasing.

Observing that $\{x_n\}_{n=0}^{\infty}$ is bounded and by Lemma 2.2, we get $\lim_{n\to\infty} x_n = 0$. Hence, the proof is complete. □

**Theorem 3.3.** *Assume that* $1 < A < A^*$ *and* $\{x_n\}_{n=0}^{\infty}$ *is a positive nonoscillatory solution of Eq.* (1). *Then there exists* $n_0 > 0$ *such that* $\{x_n\}_{n=n_0}^{\infty}$ *monotonically converges to* $\bar{x}$, *where* $A^*$ *is a constant which was derived in Lemma* 2.4, $\bar{x}$ *is the equilibrium of Eq.* (1).

**Proof.** Let $\{x_n\}_{n=0}^{\infty}$ be a nonoscillatory solution of Eq. (1). Then there exists $n_0^*$ such that $x_n \leqslant \bar{x}$ or $x_n \geqslant \bar{x}$ holds for $n \geqslant n_0^*$.

Now, we consider two cases.

*Case* 1. Assume that $x_n \leqslant \bar{x}$ for $n \geqslant n_0^*$.

From Eq. (1), we have

$$x_{n+1} = \left(1 - \sum_{j=0}^{k-1} x_{n-j}\right)(1 - e^{-Ax_n}) \geqslant (1 - k\bar{x})(1 - e^{-Ax_n}) \quad \text{for } n \geqslant n_0^* + k - 1.$$

Thus, by Lemma 2.7, we obtain

$$x_{n+1} \geqslant (1 - k\bar{x})\frac{(1 - e^{-Ax_n})}{(1 - e^{-A\bar{x}})}(1 - e^{-A\bar{x}}) > x_n \quad \text{for } n \geqslant n_0^* + k - 1.$$

Therefore, by Lemma 2.2 we have the positive solution $\{x_n\}_{n=n_0}^{\infty}$ of Eq. (1) is increasing and converges to the equilibrium of Eq. (1).

*Case* 2. $x_n \geqslant \bar{x}$ for $n \geqslant n_0^*$.

The proof of this case is just the same as Case 1, so we omit it.

This completes the proof. □

**Theorem 3.4.** *Assume that* $A = A^*$ *and* $\{x_n\}_{n=0}^{\infty}$ *is a positive nonoscillatory solution of Eq.* (1). *Then there exists* $n_0 > 0$ *such that* $\{x_n\}_{n=n_0}^{\infty}$ *increasingly converges to* $\bar{x}$, *where* $A^*$ *is a constant which was derived in Lemma* 2.4, $\bar{x}$ *is the equilibrium of Eq.* (1).

**Proof.** Let $\{x_n\}_{n=0}^{\infty}$ be a positive nonoscillatory solution of Eq. (1). Then there exists $n_0^*$ such that $x_n \leqslant \bar{x}$ or $x_n \geqslant \bar{x}$ holds for $n \geqslant n_0^*$.

*Case* 1. $x_n \leqslant \bar{x}$ for $n \geqslant n_0^*$.

The proof of this case is just the same as Case 1 of Theorem 3.2, so we omit it.

*Case* 2. $x_n \geqslant \bar{x}$ for $n \geqslant n_0^*$.

From Eq. (1), we have

$$x_{n+1} = \left(1 - \sum_{j=0}^{k-1} x_{n-j}\right)\left(1 - e^{-Ax_n}\right) \leqslant (1 - k\bar{x})\left(1 - e^{-Ax_n}\right) \quad \text{for } n \geqslant n_0^* + k - 1.$$

Thus, by Lemma 2.7, we obtain

$$x_{n+1} \leqslant (1 - k\bar{x})\frac{(1 - e^{-Ax_n})}{(1 - e^{-A\bar{x}})}\left(1 - e^{-A\bar{x}}\right) < x_n \quad \text{for } n \geqslant n_0^* + k - 1.$$

But from Eq. (1) we have

$$x_{n+1} = \left(1 - \sum_{j=0}^{k-1} x_{n-j}\right)\left(1 - e^{-Ax_n}\right)$$

and

$$\bar{x} < x_{n+k} < x_{n+k-1} < \cdots < x_{n+1} < x_n \quad \text{for } n \geqslant n_0^* + k - 1.$$

Then by Lemma 2.6 we have

$$\bar{x} < x_{n+k} < (1 - kx_{n+k-1})\left(1 - e^{-Ax_n+k-1}\right) < \bar{x}.$$

This is a contradiction. Thus Case 2 is impossible.

Hence the proof is complete. $\square$

**Theorem 3.5.** *Assume tat $A > A^*$. Then every nontrivial positive solution of Eq.* (1) *is strictly oscillatory about the equilibrium $\bar{x}$.*

**Proof.** Suppose that $\{x_n\}_{n=0}^{\infty}$ is a nonoscillatory solution of Eq. (1). Then there exists $n_0^*$ such that $x_n \leqslant \bar{x}$ or $x_n \geqslant \bar{x}$ holds for $n \geqslant n_0^*$.

Now, we consider two cases.

*Case* 1. Assume that $x_n \leqslant \bar{x}$ for $n \geqslant n_0^*$.

From Eq. (1) we have

$$x_{n+1} = \left(1 - \sum_{j=0}^{k-1} x_{n-j}\right)\left(1 - e^{-Ax_n}\right) > (1 - k\bar{x})\frac{(1 - e^{-Ax_n})}{1 - e^{-A\bar{x}}}\left(1 - e^{-A\bar{x}}\right).$$

By Lemma 2.7, we obtain $x_{n+1} > x_n$ for $n \geqslant n_0^* + k - 1$. Then $\{x_n\}_{n=n_0^*+k-1}^{\infty}$ is strictly increasing.

Suppose that $x_n \geqslant \bar{x}^*$ for $n > n_1$, where $n_1 > n_0^* + k - 1$ and $\bar{x}^*$ is a constant which was derived in Lemma 2.6. Then

$$\bar{x} > x_{n+k+1} = \left(1 - \sum_{j=0}^{k-1} x_{n+k-j}\right)\left(1 - e^{-Ax_{n+k}}\right) > (1 - kx_{n+k})\left(1 - e^{-Ax_n+k-1}\right).$$

By Lemma 2.6, we obtain $\bar{x} > x_{n+k+1} > \bar{x}$. This is a contradiction.

Suppose that $x_n < \bar{x}^*$ for $n > n_2$, where $n_2 > n_0^* + k - 1$ and $\bar{x}^*$ is a constant which was derived in Lemma 2.6.

Owing to that $\{x_n\}_{n=n_2}^{\infty}$ is bounded, $\lim_{n\to\infty} x_n \leqslant \bar{x}^* < \bar{x}$. This is also a contradiction.

*Case* 2. Assume that $x_n \geqslant \bar{x}$ for $n \geqslant n_0^*$.

The proof of this case is just the same as Case 2 of Theorem 3.3, so we omit it.

Hence, the nonoscillatory solution of Eq. (1) does not exist, which completes the proof. $\square$

## References

[1] V.L. Kocic, G. Ladas, Global Behavior of Nonlinear Difference Equations of Higher Order and Applications, Kluwer Academic, Dordrecht, 1993.